Multi-Objective Optimization: Introduction

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Solving Multiobjective Optimization Problems

Multiobjective optimization problem: MOOP

There are three components in any optimization problem:

F: Objectives

$$\text{minimize (maximize) } f_i(x_1, x_2, \cdots, x_n),\ i = 1, 2, \cdots, m$$

S: Constraints

Subject to

$$g_j(x_1, x_2, \cdots, x_n),\ ROP_j\ C_j,\ j = 1, 2, \cdots, l$$

V: Design variables

$$x_k\ ROP_k\ d_k,\ k = 1, 2, \cdots, n$$

Note :

1. For a multi-objective optimization problem (MOOP), $m \geq 2$
2. Objective functions can be either minimization, maximization or both.

A formal specification of MOOP

Let us consider, without loss of generality, a multi-objective optimization problem with n decision variables and m objective functions

$$\text{Minimize } y = f(x) = [y_1 \in f_1(x), y_2 \in f_2(x), \cdots, y_k \in f_m(x)]$$

where

$$x = [x_1, x_2, \cdots, x_n] \in X$$
$$y = [y_1, y_2, \cdots, y_n] \in Y$$

Here :

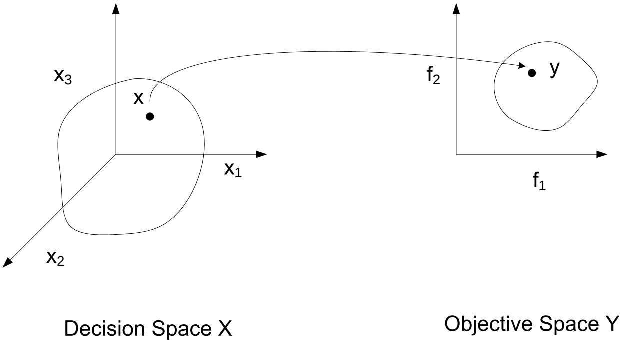
x is called decision vector

y is called an objective vector

X is called a decision space

Y is called an objective space

Illustration: Decision space and objective space



Thus, solving a MOOP implies to search for x in the decision space (X) for an optimum vector (y) in the objective space (Y).

A formal specification of MOOP (contd...)

In other words,

1. We wish to determine $\bar{X} \in X$ (called feasible region in X) and any point $\bar{x} \in \bar{X}$ (which satisfy all the constraints in MOOP) is called feasible solution.

2. Also, we wish to determine from among the set $\bar{X}$, a particular solution $\bar{x}^*$ that yield the optimum values of the objective functions.

   Mathematically,

   $$\forall \bar{x} \in \bar{X} \text{ and } \exists \bar{x}^* \in \bar{X} \mid f_i(\bar{x}^*) \leq f_i(\bar{x}),$$

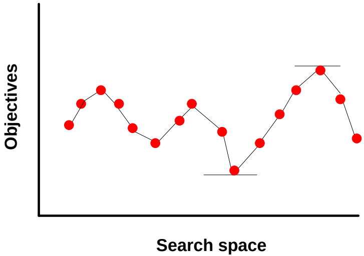
   where $\forall_i \in [1, 2, \cdots, m]$

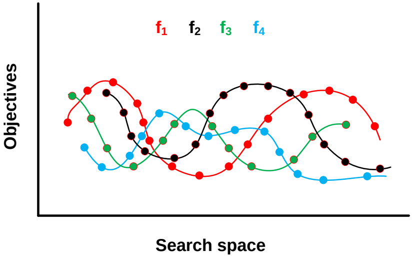
3. If this is the case, then we say that $\bar{x}^*$ is a desirable solution.

Why solving a MOOP is an issue?

- In a single-objective optimization problem, task is to find typically one solution which optimizes the sole objective function
- In contrast to single–objective optimization problem, in MOOP:
    - Cardinality of the optimal set is more than one, that is, there are $m \geq 2$ goals of optimization instead of one
    - There are $m \geq 2$ different search points (possibly in different decision spaces) corresponding to m objectives
- Optimizing each objective individually not necessarily gives the optimum solution.
    - Possible, only if objective functions are independent to their solution spaces.

Illustration: Single vs. multiple objectives

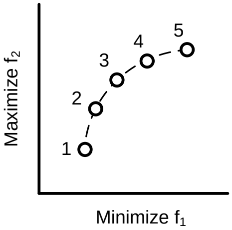




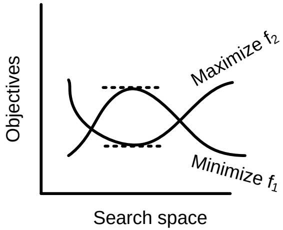
Why solving an MOOP is an issue?

- In fact, majority of the real-world MOOPs are with a set of trade-off optimal solutions. A set of trade-off optimal solutions is also popularly termed as **Pareto optimal solutions**
  - In a particular search point, one may be the best whereas other may be the worst
- Also, sometime MOOPs are with conflicting objectives
  - Thus, optimizing an objective means compromising other(s) and vice-versa.

MOOP: Trade-off and conflicts in solutions



Trade-off optimal solution

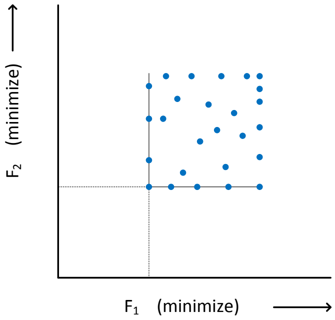


Conflicting objectives

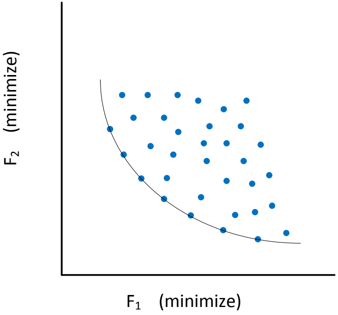
Illustration: ideal solution vs. real solution

It is observed that in many real-life problems, we hardly have a situation in which all the $f_i(\bar{x})$ have a minimum in $\bar{X}$ at a common point $\bar{x}^*$.

This is particularly true when objective functions are conflicting in their interests.



Ideal Situation

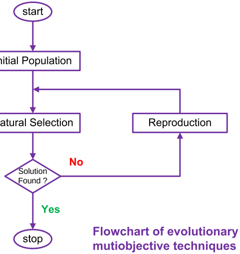


Real Situation

Multiobjective Evolutionary Algorithms

GA-based approach to solve MOOPs

- MOEA : Multi Objective Evolutionary Algorithm



Flowchart of evolutionary mutiobjective techniques

- MOEA follows the same reproduction operation as in GA but follow different selection procedure and fitness assignment strategies.
- There are also a number of stochastic approaches such as Simulated Annealing (SA), Ant Colony Optimization (ACO), Particle Swam Optimization (PSO), Tabu Search (TS), etc. could be used to solve MOOPs.

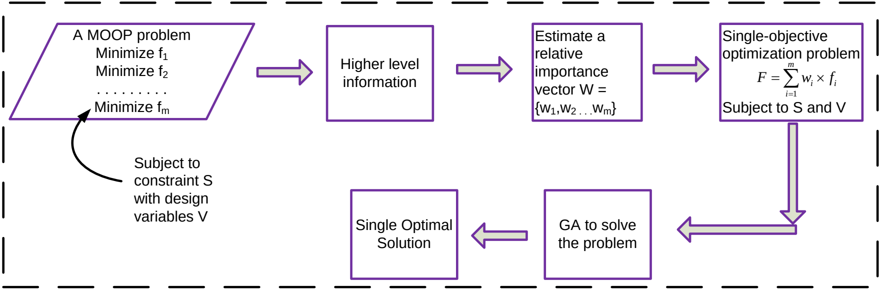
MOEA: GA-based approach to solve MOOP

- There are two board approaches to solve MOOPs with MOEA
  - A priori approach (also called preference-based approach)
  - A posteriori approach (does not require any prior knowledge)

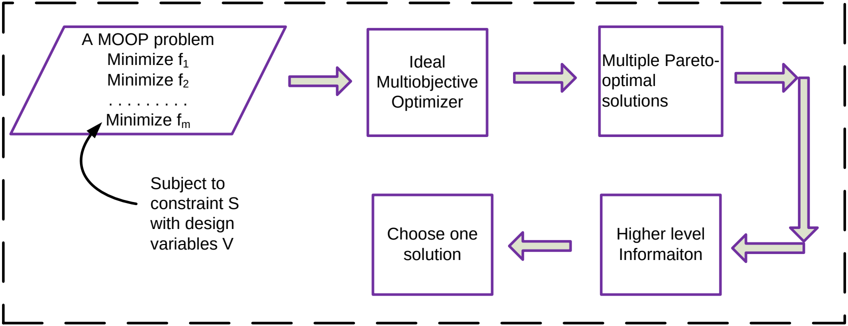
Two major problems must be addressed when a GA is applied to multi-objective optimization problems.

1. How to accomplish fitness assignment and selection in order to guide the search toward the optimal solution set?
2. How to maintain a diverse population in order to prevent premature convergence and achieve a well distributed trade-off front?

Schematic of a priori MOEA approach



Schematic of a posteriori MOEA approach



IDEAL multi-objective optimization

Here, effort have been made in finding the set of trade-off solutions by considering all objectives to be important.

Steps

1. Find multiple trade-off optimal solutions with a wide range of values for objectives. (Note: here, we do not use any relative preference vector information). The task here is to find as many different trade-off solutions as possible.
2. Choose one of the obtained solutions using higher level information (i.e. evaluate and compare the obtained trade-off solutions)

Illustration: Higher level information

Consider the decision making involved in buying an automobile car. Consider two objectives.

- **minimize Cost**
- **maximize Comfort**

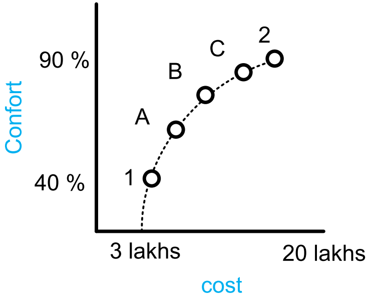


Illustration: Higher level information

- Here, solution 1 and 2 are two extreme cases.
- Between these two extreme solutions, there exist many other solutions, where trade-off between cost and comfort exist.
- In this case, all such trade-off solutions are optimal solutions to a multi-objective optimization problem.
- Often, such trade-off solution provides a clear front on an objective space plotted with the objective values.
- This front is called Pareto-optimal front and all such trade-off solutions are called Pareto-optimal solutions (after the name of Vilfredo Pareto, 1986)

Choosing a solution with higher level information

- Knowing the number of solutions that exist in the market with different trade-offs between cost and comfort, which car does one buy?
- It involves many other considerations
  - total finance available to buy the car
  - fuel consumption
  - depreciation value
  - road condition
  - physical health of the passengers
  - social status
  - After sales service, vendor's reputation, manufacturer's past history etc.

Preliminaries of MOEA

Formal specification of MOEA approach

In the next few slides, we shall discuss the above idea of solving MOOPs more precisely. Before that, let us familiar to few more basic definitions and terminologies.

1. Concept of domination
2. Properties of dominance relation
3. Pareto-optimization
4. Solutions with multiple-objectives

Solution with multiple objectives : Ideal objective vector

For each of the M-th conflicting objectives, there exist one different optimal solution. An objective vector constructed with these individual optimal objective values constitute the ideal objective vector.

Definition 1: Ideal objective vector

Without any loss of generality, suppose the MOOP is defined as

Minimize $f_m(x)$, $m = 1, 2, \cdots, M$

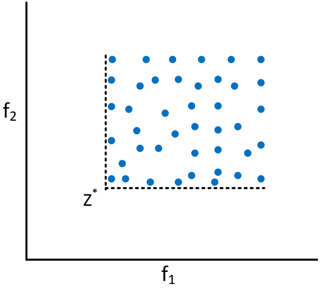
Subject to $X \in S$, where S denotes the search space.

and

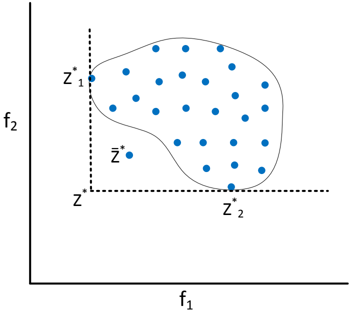
f_m^* denotes the minimum solution for the m-th objective functions, then the ideal objective vector can be defined as

$$Z^* = f^* = [f_1^*, f_2^*, \cdots, f_M^*]$$

Ideal objective vector : Physical interpretation



(A) Ideal objective vector



(B) A good solution vector should be as close to ideal solution vector

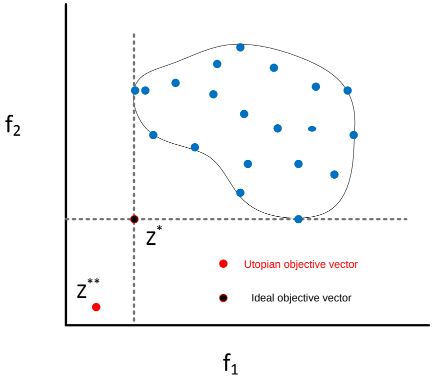
Ideal objective vector : Physical interpretation

Let us consider a MOOP with two objective functions f_1 and f_2 where both are to be minimized.

- If $z^* = f^* = [f_1^*, f_2^*]$ then both f_1 and f_2 are minimum at $x^* \in S$. (That is, there is a feasible solution when the minimum solutions to both the objective functions are identical).
- In general, the ideal objective vector z^* corresponds to a non-existent solution (this is because the minimum solution for each objective function need not be the same solution).
- If there exist an ideal objective vector, then the objectives are non-conflicting with each other and the minimum solution to any objective function would be the only optimal solution to the MOOP.
- Although, an ideal objective vector is usually non-existing, it is useful in the sense that any solution closer to the ideal objective vector are better. (In other words, it provides a knowledge on the lower bound on each objective function to normalize objective values within a common range).

Solution with multiple objectives : Utopian objective vector

Utopian objective vector corresponding to a solution which has an objective value strictly better than (and not equal to) that of any solution in search space.



Solution with multiple objectives : Utopian objective vector

The Utopian objective vector can be formally defined as follows.

Definition 2 : Utopian objective vector

A Utopian objective vector z^{**} has each of its component marginally smaller than that of the ideal objective vector, that is

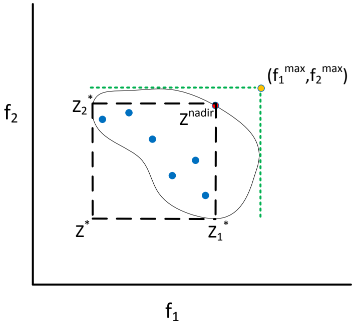
$z_i^{**} = z_i^{*} - \in_i$ with $\in_i > 0$ for all $i = 1, 2, \cdots, M$

Note :

Like the ideal objective vector, the Utopian objective vector also represents a non-existent solution.

Solution with multiple objectives : Nadir objective vector

The ideal objective vector represents the lower bound of each objective in the entire feasible search space. In contrast to this, the Nadir objective vector, denoted as z^{nadir}, represents the upper bound of each objective in the entire Pareto-optimal set (note: not in the entire search space).



Solution with multiple objectives : Nadir objective vector

Note :

z^{nadir} is the upper bound with respect to Pareto optimal set. Whereas, a vector of objective W found by using the worst feasible function values f_i^{max} in the entire search space.

Usefulness of Nadir objective vector

In order to normalize each objective in the entire range of Pareto-optimal region, the knowledge of Nadir and ideal objective vectors can be used as follows.

$$\bar{f}_i = \frac{f_i - z_i^*}{z_i^{nadir} - z_i^*}$$

Concept of domination

Notation

- Suppose, $f_1, f_2, \cdots, f_M$ are the objective functions
- x_i and x_j are any two solutions
- The operator $\lhd$ between two solutions x_i and x_j as $x_i \lhd x_j$ to denote that solution x_i is better than the solution x_j on a particular objective.
- Alternatively, $x_i \rhd x_j$ for a particular objective implies that solution x_i is worst than the solution x_j on this objective.

Note :

If an objective function is to be minimized, the operator $\lhd$ would mean the ”$<$” (less than operator), whereas if the objective function is to be maximized, the operator $\lhd$ would mean the ”$>$” (greater than operator).

Concept of domination

Definition 3 : Domination

A solution x_i is said to dominate the other solution x_j if both condition I and II are true.

Condition : I

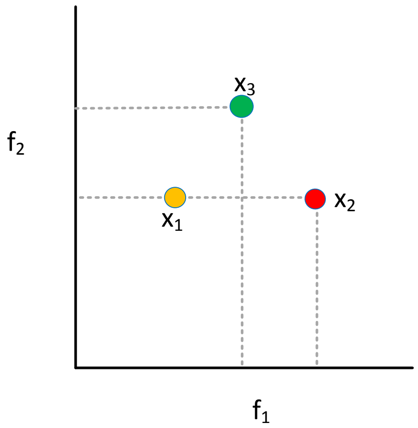
The solution x_i is no worse than x_j in all objectives. That is $f_k(x_i) \not\triangleright f_k(x_j)$ for all $k = 1, 2, \cdots, M$

Condition : II

The solution x_i is strictly better than x_j in at least one objective. That is $f_{\bar{k}}(x_i) \triangleleft f_{\bar{k}}(x_j)$ for at least one $\bar{k} \in \{1, 2, \cdots, M\}$

Illustration 1

Consider that f_1 and f_2 are two objectives to be minimized.



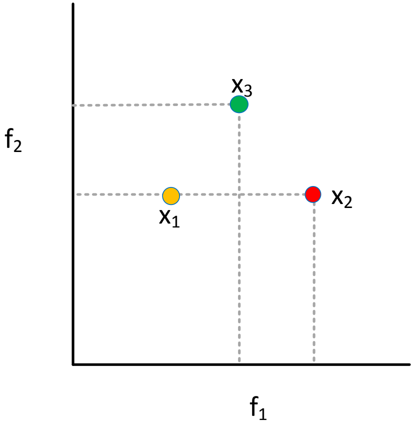
Minimize f_1
Minimize f_2

$x_1 \preceq x_2$

$x_1 \preceq x_3$ but $x_3 \npreceq x_1$

$x_2 \npreceq x_3$ as well as $x_3 \npreceq x_2$

Illustration 2



Minimize f_1
Maximize f_2

$x_1 \leq x_2$ or $x_2 \leq x_1$?

$x_1 \leq x_3$ or $x_3 \leq x_1$?

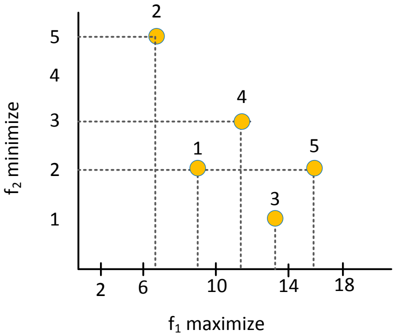
$x_2 \leq x_3$ or $x_3 \leq x_2$?

Points to be noted

Note :

- If either of the condition I and II is violated then the solution x_i does not dominate the solution x_j.
- If x_i dominates the solution x_j (it is also mathematically denoted as $x_i \leq x_j$.
- The domination also alternatively can be stated in any of the following ways.
  - x_j is dominated by x_i
  - x_i is non-dominated by x_j
  - x_i is non-inferior to x_j

Illustration 3



Here, 1 dominates 2, 5 dominates 1 etc.

Properties of dominance relation

- Definition 3 defines the dominance relation between any two solutions.
- This dominance relation satisfies four binary relation properties.

Reflexive :

The dominance relation is **NOT** reflexive.

- Any solution x does not dominate itself.
- Condition II of definition 3 does not allow the reflexive property to be satisfied.

Symmetric :

The dominance relation also **NOT** symmetric

- $x \preceq y$ does not imply $y \preceq x$.

Properties of dominance relation

Antisymmetric :

- Dominance relation **can not be** antisymmetric

Transitive :

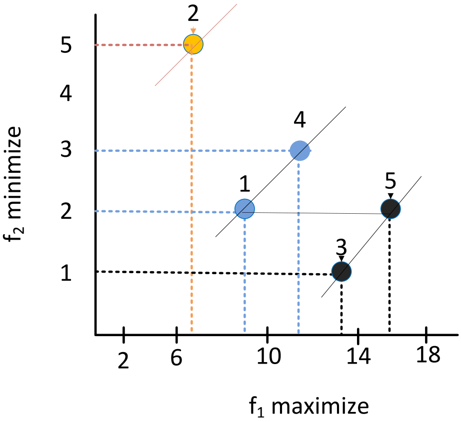
- The dominance relation is **TRANSITIVE**
  - If $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

Properties of dominance relation

Note :

1. An interesting property that dominance relation possesses is : If solution x does not dominate solution y, this does not imply that y dominates x.
2. In order for a binary relation to qualify as an ordering relation, it must be at least transitive. Hence, dominance relation qualifies as an ordering relation.
3. A relation is called partially ordered set, if it is reflexive, antisymmetric and transitive. Since dominance relation is NOT REFLEXIVE, NOT ANTISYMMETRIC, it is NOT a PARTIALLY ORDER RELATION
4. Since, the dominance relation is not reflexive, it is a STRICT PARTIAL ORDER.

Pareto optimality



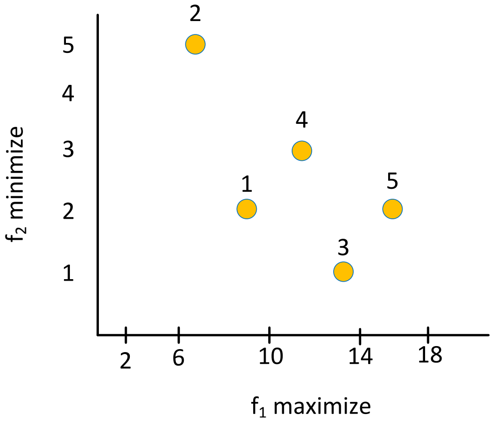
Non-dominated front

Pareto optimality

Consider solution 3 and 5.

- Solution 5 is better than solution 3 with respect to f_1 while 5 is worse than 3 with respect to f_2.
- Thus, condition I (of Definition 3) is not satisfied for both of these solutions.
- Hence, we can not conclude that 5 dominates 3 nor 3 dominated 5.
- In other words, we can not say that two solutions 3 and 5 are better.

Non-dominated set

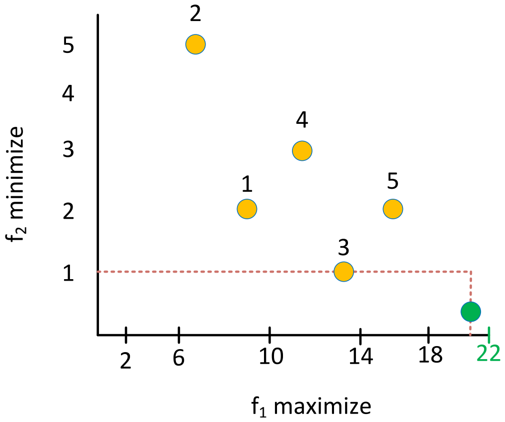


Non-dominated set

From the figure it is evident that

- There are a set of solutions namely 1, 2, 3, 4 and 5.
- 1 dominates 2; 5 dominates 1 etc.
- Neither 3 dominates 5 nor 5 dominates 3
  We say that solution 3 and 5 are non-dominated with respect to each other.
- Similarly, we say that solution 1 and 4 are non-dominated.
- In this example, there is not a single solution, which dominates all other solution

Non-dominated set: A counter example

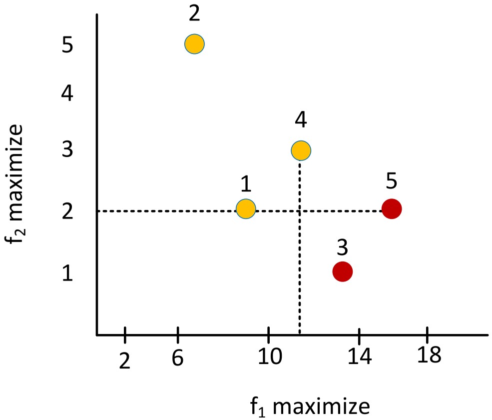


Non-dominated set

Definition 4 : Non-dominated set

Among a set of solutions P, the non-dominated set of solutions P' are those which are not dominated by any member of the set P.

Non-dominated set



Here P = {1,2,3,4,5}

Non-dominated set

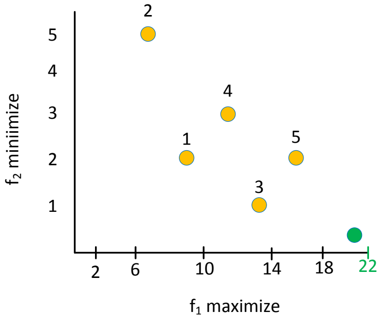
P' = {3, 5}

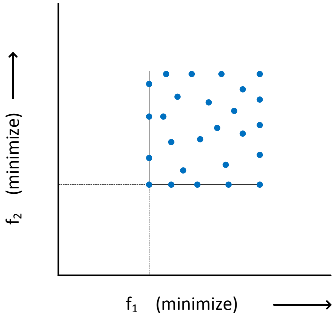
How to find a non-dominated set ?

- For a given finite set of solutions, we can perform all pair-wise comparisons.
  - Find which solution dominates
  - Find which solutions are non-dominated with respect to each other.
- Property of solutions in non-dominated set
  - $\exists x_i, x_j \in P'$ such that $x_i \not\preceq x_j$ and $x_j \not\preceq x_i$
  - A set of solution where any two of which do not dominate each other if
    - $\exists x_i \in P$ and $x_i \notin P'$ then $x_i \not\preceq x_j$ where $x_j \in P'$ for any solution outside of the non-dominated set, we can always find a solution in this set which will domnaite each other.

Some important observations

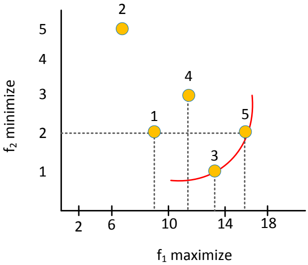
The above definition does not applicable to ideal situation.

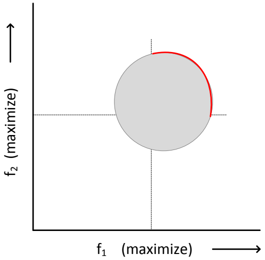


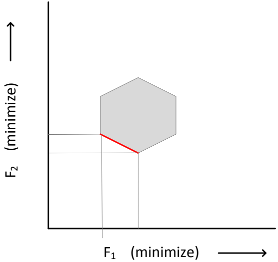


Some important observations

The non-dominated set concept is applicable when there is a trade-off in solutions.



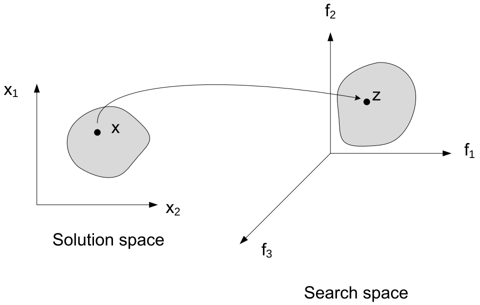




Pareto optimal set

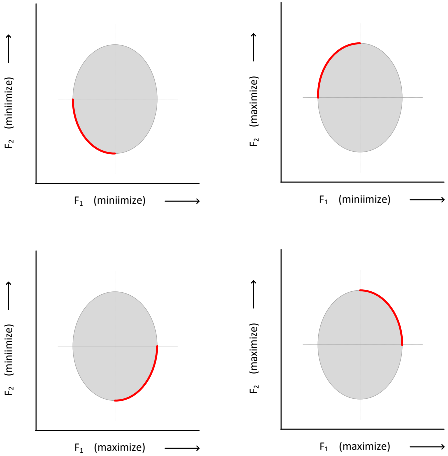
Definition 5: Pareto optimal set

When the set P is the entire search space, that is $P = S$, the resulting non-dominated set P' is called the Pareto-optimal set.



Examples: Pareto optimal sets

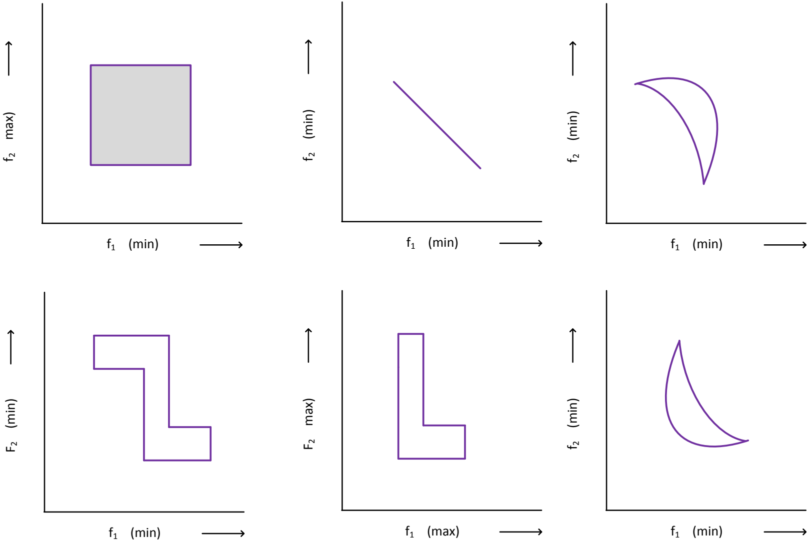
Following figures shows the Pareto ooptimal set for a set of feasible solutions over an entire search space under four different situations with two ojective functions f_1 and f_2.



Pareto optimal fronts

In visual representation, all Pareto optimal solutions lie on a front called Pareto optimal front, or simply, Pareto front.

Examples


f2 max)
f1 (min)
f2 (min)
f1 (min)
f2 (min)
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F2 max)
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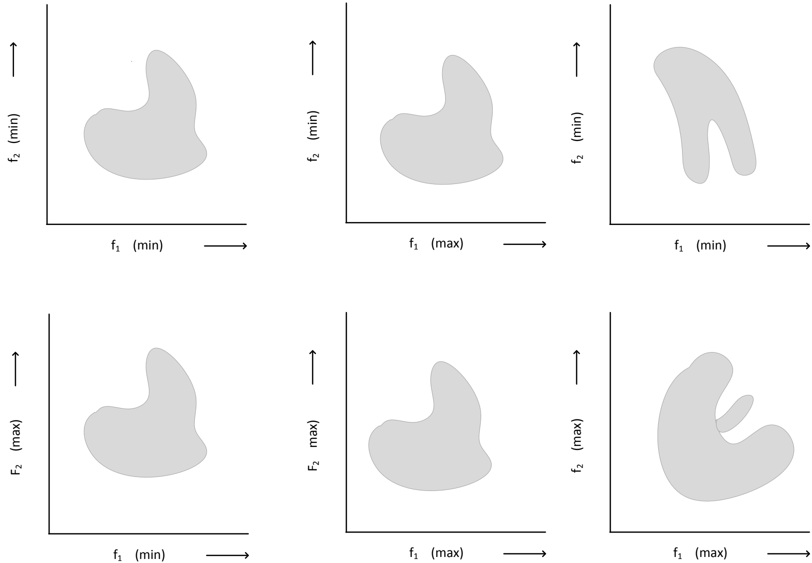
f1 (min)

Examples



f_2 (min)
f_1 (min)
f_2 (min)
f_1 (max)
f_2 (min)
f_1 (min)
F_2 (max)
f_1 (min)
F_2 max)
f_1 (max)
f_2 (max)
f_1 (max)

Few good articles to read.

1. ”An Updated Survey of GA Based Multi-objective Optimization Techniques” by Carles A Coello Coello, *ACM Computing Surveys*, No.2,Vol. 32, June 2000.
2. ”Comparison of Multi-objective Evolutionary Algorithm : Empirical Result” by E. Zitzler, K.Deb, Lother Thiele, *IEEE Transaction of Evolutionary Computation*, No.2, Vol.8, Year 2000.

Any questions??